“Unfortunately, no one can be told what the Matrix is. You have to see it for yourself.”
-- Morpheus

Primary concepts:
- Row and column operations,
- Matrix and its transpose,
- Symmetric matrices,
- Trace of a matrix,
- Inverse of a matrix, singular vs. non-singular matrices


By now you are comfortable with basic matrix operations (if not, stop! Go back! Do more practice!)

In this lab, there are a few special ops that we would like to investigate.

**Permutations : Swapping columns**

Given a matrix  \( A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{bmatrix} \), suppose we needed to change the order of the columns to  \( B = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 8 \\ 9 & 3 & 27 \end{bmatrix} \), ie, swap columns 1 and 2. How can this idea be represented as a simple operation in matrix terms?

The answer is a *permutation matrix*. Consider the matrix  \( P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

When we take the product of the matrices  \( A \) and  \( P_{12} \) (note the order), what do we get?

For these permutation matrices, we will use the convention that the subscripts of  \( P \) represent the columns that are being swapped, so that  \( P_{12} \) represents a swap of columns
one and two. The permutation comes from the identity matrix I: just swap the columns of I that correspond to the desired permutation of A. It should also be apparent that all these permutations are reversible: What is $P_{12}P_{12}$?

Write $P_{13}$ and $P_{23}$ and verify that they perform the desired permutations of A.

What would a permutation matrix that was designed to swap all three columns in just one step look like?

Write a P matrix to first swap columns 1 and 3 and then columns 2 and 3 of the resulting matrix such that

$$C = AP_{132} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 2 & 4 \\ 27 & 3 & 9 \end{bmatrix}$$. Note this is easiest to visualize by doing in two steps with two separate $P_{jk}$ matrices; then just multiply the two P matrices together!

Recall that matrix multiplication is associative, but NOT commutative.

Suppose we agree that I is also a permutation matrix (it just doesn’t do anything: $I = P_{11}$ or $I = P_{22}$ or $I = P_{33}$). How many total permutation matrices (including the two-column swappers) are there for the 3x3 matrix A? Do the same count for a 2x2 and a 4x4 matrix of your choice. What pattern do you see emerging?

**Swapping rows: permutations again!**

Suppose we want to obtain

$$D = \begin{bmatrix} 2 & 4 & 8 \\ 1 & 1 & 1 \\ 3 & 9 & 27 \end{bmatrix}$$

from the original matrix A. We have swapped the first two rows. Do we need a brand new permutation matrix? Let’s use the same $P_{12}$ from the column swap, but this time put the permutation matrix to the left of A and multiply. We certainly do not expect the product $P_{12}A$ to equal the product $A P_{12}$. Does $D = P_{12}A$?

What is the result of $P_{21}P_{12}A$? What does this say about any product $P_{jk}P_{kj}$ and the relationship between $P_{jk}$ and $P_{kj}$?

Try a double row swap; use the appropriate P matrix and verify that it does what you expect.
**Transpose of a matrix**

Consider the rectangular 3x2 matrix \( R = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 1 \end{bmatrix}. \)

We define the transpose of \( R \) to be the matrix obtained when the columns of \( R \) are written as rows:

\[
R^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}
\]

Note that the transpose operation is written with a superscript capital \( T \). This operation produces the following relationship between the elements of \( A \) and \( A^T \):

\[(A^T)_{jk} = A_{kj}\]

The transpose operation is immediately reversible: \((A^T)^T = A\)

Symmetric matrices (which have to be square) are their own transpose: \( S = S^T \). This is obviously true for the Identity matrix.

For any rectangular matrix \( R \), the product \( R^T R \) is always symmetric (and therefore square). If \( R \) is \( m \times n \), \( R^T \) is \( n \times m \) and the product \( R^T R \) is \( n \times n \). What can you say about the product \( R R^T \)?

We can prove the symmetry of \( R^T R \) simply by taking its transpose:

\[
(R^T R)^T = R^T (R^T)^T = R^T R.
\]

Do the same for \( R^T R \).

Go back and look at one of the permutation matrices and the permutation that undoes it. What is the relationship between ‘invert’ and ‘transpose’ for permutation matrices? A matrix that undoes the result of a prior matrix operation is an inverse matrix (much more about this later).

Note the “Properties of the Transpose” listed in Dawkins’ chapter on Properties of Matrix arithmetic, including some proofs.

**Trace**

The trace of a square matrix is the sum of its diagonal elements, indicated \( tr(A) \).

Interesting property of trace: \( tr(A) = tr(A^T) \). And that is the only reason it is mentioned here. More on trace later.
Problems

Prove or disprove (by counterexample) the following generalizations about matrix transposition: For two conformable matrices $A$ and $B$ of any rectangular size and shape,

1. $(A + B)^T = A^T + B^T$

2. $(AB)^T = B^T A^T$

Why do the matrices being multiplied change place under the transpose operation? Start by noting that if $A$ is $m \times p$, a conformable $B$ must be $p \times n$. Then $A^T$ is $p \times m$ and $B^T$ must be $n \times p$. For there to be a product $A^T B^T$, $m = n$. But $A$ and $B$ are rectangular!

3. Suppose $A$ and $B$ are both $n \times n$ symmetric matrices. Is matrix multiplication of $A$ and $B$ commutative? Why or why not?

Note: A proof is a complete chain of logical statements relating a claim or assertion to a verifiable conclusion. It is therefore never enough to just supply a single example when asked for a proof. However, a single counterexample is valid to show that a claim is not generally true. Many valid proofs begin by assuming that what is to be proved is false and arrive at a logical contradiction. So what was assumed to be false must in fact be true. And when we are done with a proof, we say QED (quite easily done).

Inverse of a matrix (for starters)

Definition (for you squares only): If $A$ is a square matrix and its inverse matrix $A^{-1}$ exists, then

$$A^{-1} A = I = A A^{-1}.$$  

Multiplication of a square matrix by its inverse must produce the identity, whether on the left and on the right (commutative). What is the inverse of a row swap matrix?

One of the problems that will haunt us throughout this course is the question of what is necessary for a matrix to have an inverse — and then how to find that inverse, of course!

In the $2 \times 2$ case, there is a simple formula for finding the inverse. In case you don’t remember the $2 \times 2$ formula,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$  

The quantity $ad - bc$ is known as the determinant of the $2 \times 2$ matrix, an old friend (ha!) from Algebra 2.
For anything bigger than 3x3, you do not want to have to invert the matrix using a formula. In this course, calculating the determinant of a large matrix is a method of last resort.

Example

\[
A = \begin{bmatrix}
1 & 3 \\
2 & 7
\end{bmatrix}
\]
has inverse \[A^{-1} = \begin{bmatrix}
7 & -3 \\
-2 & 1
\end{bmatrix}.
\]

Conveniently, the determinant of A is 1 and our formula just flips the signs of \(b\) and \(c\) and swaps the places of \(a\) and \(d\).

We can also obtain this inverse by solving the four equations:

\[
\begin{align*}
x_1 + 3x_3 &= 1 \\
x_2 + 3x_4 &= 0 \\
2x_1 + 7x_3 &= 0 \\
2x_2 + 7x_4 &= 1
\end{align*}
\]

or the two matrix equations

\[
\begin{bmatrix}
1 & 3 \\
2 & 7
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
1 & 3 \\
2 & 7
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_4
\end{bmatrix} = \begin{bmatrix}
0 \\
1
\end{bmatrix},
\]

which come from the expression \(AA^{-1} = I\). Be sure you see how we arrived at those equations!

Note that in this example, the inverse is written as

\[
A^{-1} = \begin{bmatrix}
x_1 & x_2 \\
x_3 & x_4
\end{bmatrix}
\]

and we just split the matrix into column vectors.

**So you think that inverses apply only to square matrices?**

Not so fast! Rectangular matrices can have left and right inverses: read on, but slowly

If \(A\) is \(m \times n\), then \(L_{n \times m}\) is a left inverse for \(A\) if and only if \(LA = I_n\)

Similarly, \(R_{n \times m}\) is a right inverse for \(A\) if and only if \(AR = I_m\)

Claim: If \(A\) has both an \(L\) and an \(R\), then \(L = R\).

Why? Start with \(LA = I\) and \(AR = I\). Then \(L = LI = L(AR) = (LA)R = IR = R\)
Example

Find L and R (if they exist) for \( A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \)

To begin, \( LA = I = AR \). Thus L must be 2x3, as must R.

Let \( L = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \) and generate some equations:

\[
LA = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = I \implies \begin{align*}
a + 2b + 4c &= 1 \\
d + 2e + 4f &= 1 \end{align*}
\]

Oh no, only 4 equations, with 6 unknowns!

But we just showed that if both L and R exist, \( L = R \).

So

\[
AR = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = I \implies \begin{align*}
a + d &= 1 \\
b + e &= 0 \\
c + f &= 0 \\
2a + 3d &= 0 \\
2b + 3e &= 1 \\
2c + 3f &= 0 \\
4a + d &= 0 \\
4b + e &= 0 \\
4c + f &= 1 \end{align*}
\]

9 more equations, for a total of 13, with 6 unknowns!!! We can only hope that some of the equations are redundant, so that a unique solution can be found. If not, there are no inverses for \( A \). Find out!

Questions: To have both an L and an R, must \( A \) be a square matrix?

If L is a left inverse for rectangular matrix \( A \), prove that \( L^T \) is a right inverse for \( A^T \).

If R is a right inverse for rectangular matrix \( A \), prove that \( R^T \) is a left inverse for \( A^T \).

An invertible matrix is also called a non-singular matrix. A matrix that has no inverse is said to be singular or have a singularity.

What’s important about the singular/non-singular question?

For two matrices \( A \) and \( B \) and the appropriately sized 0 matrix, if \( AB = 0 \), then either \( A = 0 \), \( B = 0 \) or both \( A \) and \( B \) are singular matrices.

If \( Ax = b \) and \( A \) has inverse \( A^{-1} \), then \( A^{-1}Ax = A^{-1}b \) or \( x = A^{-1}b \).
Thus if $A$ has an inverse, there is at least one solution $x$ for every $b$. Further, if $A$ has an inverse and $Ax = 0$, the only solution is the trivial solution $x = 0$.

Is $x = A^{-1}b$ a good way to solve $Ax = b$? Maybe yes, maybe no.

**Some properties of matrix inverse**

Assume that $A$ and $B$ are both square, nonsingular matrices

1. $(AB)^{-1} = B^{-1}A^{-1}$  
   A proof of this important property has two distinct steps.
   
   1. Start with $B^{-1}A^{-1}$ and multiply on the right by $AB$.
   2. Start with $B^{-1}A^{-1}$ and multiply on the left by $AB$.

   Finish the proof. If these two products can be shown to be $I$, what have we shown? How does that satisfy the definition of the matrix inverse? Is that a sufficient proof?

2. $(A^{-1})^{-1} = A$

   In the same manner as the first property, if the statement is true, $(A^{-1})^{-1}A^{-1} = I$. Finish the proof.

3. $(A^T)^{-1} = (A^{-1})^T$

   Using the same approach, if this property is true, $(A^{-1})^T(A^T) = I$ and $(A^T) (A^{-1})^T = I$. Finish the proof.

4. If $A$, $B$ and $C$ are $p \times p$ nonsingular matrices then both of the following must be true:
   a. the product $ABC$ is nonsingular and b. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. Prove it!

Much more on inversion later!
Loading the Matrix: Augmentation

It is often convenient to create an augmented matrix from a given matrix A and a vector b by tacking the values of b onto the right of A, creating a temporary rectangular matrix. The symbol for the augmented matrix is sometimes \( A|b \):

\[
A | b = \begin{bmatrix}
1 & 2 & 2 & 4 \\
1 & 3 & 3 & 5 \\
2 & 6 & 5 & 6
\end{bmatrix}
\]

Any operation we do to the augmented matrix \( A|b \) is identical to doing the same operation to the right and left hand side of the equation \( Ax = b \).

In some cases, it is also possible to augment a matrix with another matrix:

\( A|B \) is only valid if A and B have the same number of rows.

See Dawkins’ section on Special Matrices for some special matrices.

Practice Problems

a. If matrix A is 5x3 and the product AB is 5x7, what is the size of B?

b. Let \( A = \begin{bmatrix}
2 & 5 \\
-3 & 1
\end{bmatrix} \) and \( B = \begin{bmatrix}
4 & -5 \\
3 & k
\end{bmatrix} \). How many values of k, if any, will make the matrix products \( AB = BA \)?

c. True/false (justify your answer):
   1. If A, B and C are arbitrary 2x2 matrices. The columns of A and B are represented as \( a_1 \), \( a_2 \) and \( b_1 \), \( b_2 \), respectively, then \( AB = \begin{bmatrix} a_1 b_1 & a_2 b_2 \end{bmatrix} \)
   2. Each column of \( AB \) is a linear combination of the elements in a column of B using weights from the corresponding rows of A.
   3. \( AB + AC = A(B + C) \)

Practice Answers: a. 3x7 
b. one (what is it?) 
c. 1 = false: matrix multiplication is row by column, not column by column 
   2 = true: follows from the correct statement of 1 
   3 = true: suppose \( D = B+C \), can show that if \( AB + AC \) is not \( AD \) a contradiction develops.
Problem Set

1a. Construct a random 4x4 matrix $A$ and test whether $(A + I)(A – I) = A^2 – I$. What is the meaning of $A^2$? Try at least three different matrices, including both symmetric and non-symmetric cases.

b. Test $(A + B)(A – B) = A^2 – B^2$ in the same manner. Prepare a report of your findings, sufficient to document the claim that you have discovered binomial factorization of matrices. Be sure you address the commutivity issue!

2. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}$; find $A^{-1}$ by hand. Use $A^{-1}$ to solve $Ax = b$ for the following values of $b$: $b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

3. Suppose $A$, $B$ and $C$ are all invertible $n \times n$ matrices. Show that $ABC$ is also invertible by producing matrix $D$ such that $(ABC)D = I = D(ABC)$.

4. Solve the equation $AB = BC$ for $A$, assuming $A$, $B$ and $C$ are square and $B$ is invertible. Then find an equality for $C$ in terms of $A$, $B$ and $B^{-1}$.

5. Suppose $P$ is invertible and $A = PBP^{-1}$. Solve for $B$ in terms of $A$.

6. Suppose $A$, $B$ and $X$ are $n \times n$ matrices with $A$, $X$ and $A – AX$ invertible and suppose $(A – AX)^{-1} = X^{-1}B$. Explain why $B$ must be invertible and use this information to solve for $X$. 