Wave equation examples


The function \( u(x, t) \) is a solution to the classical one-dimensional wave equation if it satisfies the PDE

\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}
\]

The wave function \( u \) is the amplitude of the wave as a function of time and position. The constant \( v \) is the wave’s velocity in the \( x \) direction.

For a derivation of the wave equation, see http://hyperphysics.phy-astr.gsu.edu/hbase/waves/waveq.html -- Physics!

Since the wave equation is a linear second order PDE, given any two twice-differentiable functions of a single variable (call them \( f_1 \) and \( f_2 \)), the most general solution is

\[
u(x, t) = f_1(x + vt) + f_2(x – vt).
\]

That’s almost all there is to it! (except for the details – ah, the details).

This was first noted by Jean D’Alembert, 18th century French mathematician and bon vivant. The plus/minus signs in \( x + vt \) and \( x – vt \) indicate the direction of wave travel: \( f_2(x – vt) \) is traveling to the right and \( f_1(x + vt) \) is traveling to the left. How can you remember that? Think of surfing a wave: you want to stay in the same relative position, riding the wave crest. As time goes on (\( t \) increases, you and the wave both move to the right (your \( x \) position increases). In order to keep the same relative point on the wave function, you’d better be surfing \( f(x – vt) \). Was D’Alembert a surfer? With that hair? Not likely.

A leprechaun caught surfin’ the cosine wave off Malibu. As \( t \) and \( x \) increase, he rides \( x – vt \), staying at the same wave height.
Example Let \( f_1(x + vt) = \cos(x + vt) \) and \( f_2(x - vt) = 1 - (x - vt)^2 \).

Then \( u(x, t) = 1 - (x - vt)^2 + \cos(x + vt) \) is shown to be a wave function if it satisfies the wave equation. Show that it fits the PDE. Graph the function in \( x \) and \( t \) (especially using \textit{Animate} to plot the function of \( x \) and animate it in time).

The form of the solution to the wave equation is determined by both the initial conditions (what is the value of \( u \) when \( t = 0 \)?) and the boundary conditions (what must the wave function do at end points of the domain?).

No boundaries: traveling waves on a very long string

D’Alembert’s analysis of wave functions leads to several important results.

First, we analyze the wave equation with ICs only. Let’s write the wave equation as

\[
\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}
\]

and the wave function \( u(x, t) = f_1(x + vt) + f_2(x – vt) \).

a. We are given an initial displacement \( u(x, 0) = f(x) \) and initial velocity \( u_t(x, 0) = 0 \). This is a guitar string plucked with finger or pick (although D’Alembert would have studied the harpsichord).

Applying the second (velocity) IC first, take the required derivatives of \( u \):

\[
u_t(x, 0) = 0 = v f_1' (x) - v f_2' (x) \quad \text{or} \quad f_1' = f_2'.
\]

We integrate this directly to obtain \( f_1 = f_2 + C \) and therefore

\[
u(x, t) = f_1(x + vt) + f_1(x - vt) - C
\]

Applying the first IC \( u(x, 0) = f(x) \),

\[
f(x) = f_1(x) + f_2(x) = 2 f_1 - C \quad \text{or} \quad f_1 = \frac{f(x) + C}{2}
\]

Combine:

\[
u(x, t) = \frac{1}{2} f(x + vt) + \frac{1}{2} f(x - vt) + \frac{1}{2} f(x + vt) + \frac{1}{2} f(x - vt) - C
\]

The solution function is therefore always a sum (superposition) of \( \frac{1}{2} \) of the function that describes the shape of the string pluck; the constant cancels.
b. An initial velocity $u_t(x,0) = g(x)$ is given and the initial displacement $u(0,t) = 0$. This is a piano string struck by pressing a key or the very cool instrument known as a hammered dulcimer.

Use the method above to eliminate $f_2$:

$$u(x,t) = f_1(x-2vt) + f_1(x+2vt)$$

and then show that $f_1(x) = -\frac{1}{2v} \int_0^x g(s) ds$, where $s$ is a dummy variable that disappears upon integration.

Combine to obtain $u(x,t) = \frac{1}{2v} \int_{x-2vt}^{x+2vt} g(s) ds$

c. Combination of conditions: $u(0,t) = f(x)$ and $u_t(x,0) = g(x)$

Combined ICs yield combined solutions known as D’Alembert’s Formula:

$$u(x,t) = \frac{1}{2} f(x-2vt) + \frac{1}{2} f(x+2vt) + \frac{1}{2v} \int_{x-2vt}^{x+2vt} g(s) ds$$

Example: Apply D’Alembert’s Formula to form the wave function given by the initial condition (pluck) $u(x,0) = \frac{1}{2} e^{-x^2}$ with $v = 4$.

We see immediately that $u(x,t) = \frac{1}{2} e^{-(x+2vt)^2} + \frac{1}{2} e^{-(x-2vt)^2}$, as illustrated below. The initial pulse starts at $x = 0$ and splits in two, one traveling left, the other traveling right. Since there are no boundaries, the pulses continue moving away from each other … forever.
Example

Suppose the ICs are \( u(x, 0) = \frac{1}{2} e^{-x^2} \) and \( u_t(x, 0) = -e^{-x^2} \) for \( v = 4 \).
Use the D’Alembert Formula to find the wave function \( u(x,t) \).

Once the functions \( f \) and \( g \) are defined, this statement will find values of \( u(x,t) \) using the D’Alembert Formula:

\[
\begin{align*}
\text{Woddya know? A stamp!} \\
\text{http://jeff560.tripod.com/images/dalemb.jpg}
\end{align*}
\]

Now that we know the form of the solutions, we can look at some BVPs

We’ll start with a one-sided boundary: Suppose a horizontal string is tied at one end (say \( x = 0 \)), where it cannot move and thus \( u(0,t) = 0 \) and \( u_t(0,t) = 0 \)

In order to prevent any displacement at the bound end, a “reflection” will be generated – a wave of opposite polarity will originate at the boundary. When the incoming wave and the reflected wave are superimposed, they cancel.

Example: The pulse begins at \( x = 3 \) so that \( u(x, 0) = \frac{1}{2} e^{-(x-3)^2} \).
We must form a function that extends a negative of our wave function into \( x < 0 \) so that the sum of the wave displacement is 0.

In general, this can be done by turning the wave function into an odd functions by an ‘odd flip:’ \( y(x) \) is redefined as \( -y(-x) \) for \( x < 0 \). The most compact way to do this (but certainly not the only way) is as follows:

\[
u(x,t) = \frac{1}{2} [\text{Sign}[x + vt] f(Abs[x + vt]) + \text{Sign}[x - vt] f(Abs[x - vt])]
\]
This really creates two wave functions – the one we see from the starting point of the pulse and a mirror image (in reversed polarity) starting from \( x < 0 \).

Each pulse splits, with one half moving left and the other right.

When the ‘real’ wave and the mirror image pass through each other at the boundary, they cancel out.

But we are only interested in what happens with \( x > 0 \), so it looks like the original pulse is reflected at \( x = 0 \); then both pulses move to the right.

Verify that the wave function \( u(x,t) = 0 \) at a reflecting boundary for all values of \( t \).

It is also possible to reflect at the right hand boundary.
Pulse moving right, striking boundary at $x = 15$.

Reflected pulse (reverse polarity) now moving to the left.

Reflections at both boundaries are also possible – but require additional trickery.

Pulse begins at $x = 5$, $t = 0$

Pulse splits, parts move left and right, about to strike boundaries at $x = 0$ and $x = 10$. 
Reflected pulses now moving back towards $x = 5$

This required conditional function definition using /; Mathematica’s conditional definition operator.

\[
pulse2[x, z_0, \text{left}_, \text{right}_] := e^{-(x-z_0)^2} /; \text{left} < x < \text{right} \\
pulse2[x, z_0, \text{left}_, \text{right}_] := -e^{-(x-z_0)^2} /; x < \text{left} \quad (* \text{left is presumed to be zero *} *) \\
pulse2[x, z_0, \text{left}_, \text{right}_] := -e^{-(x_\text{right}+z_0)^2} /; x \geq \text{right} \\
\]

\[
\text{Plot}[.5 \ pulse2[x + t, 5, \text{left}_, \text{right}_] + .5 \ pulse2[x - t, 5, \text{left}_, \text{right}_], \{x, \text{left}_, \text{right}_\}, \text{PlotRange} \rightarrow \{-1, 1\}, \text{AxesLabel} \rightarrow \{x, "u[x,t]"\}, \text{PlotLabel} \rightarrow t] \\
\]

Values for \text{left} and \text{right} (the x position of the boundaries) can be explicitly assigned prior to the Plot[ ] or set with a list replacement within the Plot[ ].

We have defined the velocity of the wave as the value $v$ in the wave equation. What is the derivative $u_t(x,t)$ represent in physical terms?

What happens if we use a continuous cosine $u(x, t) = \frac{1}{2} \cos(x - vt) + \frac{1}{2} \cos(x + vt)$ instead of a discrete pulse? Try it!

A more general means of finding a wave function when there are boundary conditions involves the technique of Separation of Variables. Work that lab before continuing below.

See [http://www.math.duke.edu/education/ccp/materials/engin/wave/index.html](http://www.math.duke.edu/education/ccp/materials/engin/wave/index.html). Work through all parts of this webpage and answer the questions in the summary. We will get to Fourier Series solutions after a while; for the moment, just think of them as an approximation to the given function formed by adding sines and cosines.

We will use Separation of Variables to consider each of the following cases, each specified by a different set of boundary conditions.
See notes in wave equationBVP.pdf

1. String of length L tied at both ends (standing waves)
   
   Boundary conditions \( u(0,t) = u(L,t) = 0 \). An initial amplitude \( u(x,0) \) or particle velocity \( u_t(x,0) \) or some combination of these ICs may be specified.

For an excellent animation of a standing wave on a string, see http://galileo.phys.virginia.edu/classes/152.mf1i.spring02/forces%20on%20wave.swf

2. Tube of length L open at one end (standing waves)
   
   Boundary conditions \( u(0,t) = 0 \) \( u(L,t) = A \), for an amplitude value A.

3. String of length L tied at one end and shaken with amplitude A from the other end (traveling waves)
   
   Boundary conditions \( u(0,t) = 0 \) \( u(L,t) = A \). An initial position or velocity must be specified.

The vibrating drumhead (circular case)

The two-dimensional wave equation can be expressed in polar coordinates.

\[
\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \text{becomes} \quad \frac{\partial^2 U}{\partial t^2} = \frac{1}{c^2} \left( \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} \right).
\]

Here, \( u(x, y, t) \) is a amplitude displacement function in rectangular coordinates and \( U(r, \theta, t) \) is the displacement function transposed into polar coordinates. Good news: It is still variables separable!

Suitable boundary conditions might be fixed edges at \( r = 1 \) and an initial displacement or velocity at the center.

![Diagram of vibrating drumhead](image1)

![Diagram of vibrating drumhead](image2)

![Diagram of vibrating drumhead](image3)